are fulfilled in correspondence with the dispersion relations (6). In this case a decay instability [5] takes place. From analysis of the dispersion curves (6), it follows that fulfillment of conditions (11) and (12) is possible for $q > 2k_*$. In particular if $k_0 < q_1 < k_*$ (let $q_1 \leq q_2$), it is possible to consider that $\omega_1(q_1) \approx \omega_0 = c^2(1 - \mu^2)k_0^2$ for the flexure branch $[\omega_1(q) \equiv \Omega_2(q)$ and also $\omega_2(q) \equiv \Omega_2(q)]$. Then by using (10), we find from (11) and (12) that $\omega_2 = \omega - \omega_0$, $q_2 = (\omega^2 - 2\,\omega\omega_0)^{1/4}(ch)^{-1/2}$, and $q_1 = q - q_2$. If we take $q < 2k_*$ for the same region of q_1 values and consider that $\omega(q) < \omega(2k_*) \approx 2\omega_0$, we obtain $\omega_1 + \omega_2 > \omega$. For values $q_1 < k_0$, such that $\omega_1(q_1) < \omega_0$, Eqs. (11) and (12) can be fulfilled only if $q \gg k_*$. The analogous situation arises in the case where the first longitudinal mode is selected for the first wave $[\omega_1(q) \equiv \Omega_1(q)]$.

Finally by summarizing the results obtained, we can conclude that a packet of flexure waves in a cylindrical shell is unstable in the following regions of wave numbers: $0 < q < k_b$, $k_* - a < q < q_0$, and $q > 2k_*$.

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ANALYSIS OF THRESHOLD-FREE FRACTURE OF MATERIALS ON REFLECTION OF A COMPRESSION PULSE FROM A FREE SURFACE

A. V. Utkin

UDC 532.593

Investigations of spalling phenomena yield information about the resistance of materials to fracture under microsecond loads. The most reliable and informative method is the method in which the velocity of a free surface is recorded continuously [1]. Figure 1 shows results of such experiments for plexiglass and rubber (curves 1 and 2) [2, 3]. The character of the spalling of plexiglass is typical for solids. After the shock wave reaches the free surface, the velocity profile repeats the shape of the compression pulse in the sample. When the tensile stresses reach a critical value, the material fractures, the stresses in the fracture zone decrease, and there appears a compression wave, which reaches the surface in the form of a spallation pulse. The subsequent velocity oscillations are due to the circulation of compression and rarefaction waves in the spalled plate. The fracture stress is determined by the difference between the maximum velocity of the surface and the velocity in front of the spallation pulse [1].

A fundamentally different result was obtained for rubber. According to Fig. 1, in this case the velocity decreases monotonically and characteristic oscillations are not observed. Since there is no clearly pronounced spallation pulse, there arises the question of how the fracture process should be characterized. If the strength of rubber were negligible, then after the shock wave reaches the free surface the velocity of the surface would remain constant. The dashed line in Fig. 1 shows the velocity profile, constructed assuming that the strength of rubber is high. The experimentally observed time dependence differs from the extreme cases by high and negligibly low dynamic tensile strength. The sample remaining after this experiment did not exhibit any clear indications of fracture.

It is known [4, 5] that rupture of elastomers is preceded by formation of microscopic nonuniformities in the sample, which starts at stresses much lower than the rupture stresses. The formation of nonuniformities is in itself still not fracture. Thus, in tests under tri-

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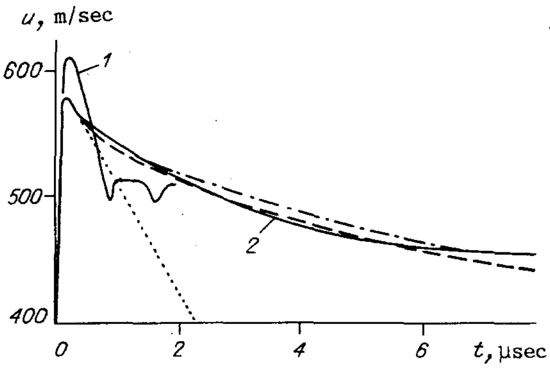

Fig. 1

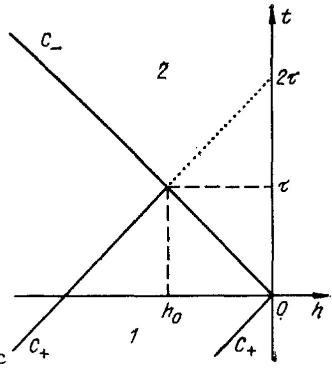

Fig. 2

axial stretching of vulcanizates of natural rubber [5], pores appeared under stresses of 1-3 MPa and very small deformations. After this, the samples underwent further deformation of several hundreds of percent, accompanying an increase (with low modulus) of the tensile stresses. For this reason, rubber is an example of a medium which has zero fracture threshold but can nonetheless withstand quite high tensile stresses (of the order of 100 MPa). The method for determining these stresses in a situation similar to that presented in Fig. 1 (curve 2) is unclear.

In this paper we analyze shock-wave processes in media with zero fracture threshold with reflection of a compression pulse from the free surface and we establish a relation between the tensile stresses and the experimentally measured velocity profile.

Formulation and Solution of the Problem. We now consider in the acoustic approximation the evolution of a triangular compression pulse after the pulse is reflected from the free surface of the sample, which fractures under negative pressure. Suppose that fracture starts with zero tensile stresses and is described by a specific pore volume v_{pore}. The total specific volume of the medium is equal to the sum of v_{pore} and the specific volume of the continuous component v_{con}: $v = v_{pore} + v_{con}$. We employ the simplest possible fracture kinetics: the rate of change of v_{pore} is a linear function of the pressure P and is equal to zero if $P > 0$ and $v_{pore} = 0$. The system of equations of hydrodynamics, closed by the kinetic equations and the equation of state, has the following form in Lagrangian variables:

$$\frac{\partial v}{\partial t} - \frac{1}{\rho_0}\frac{\partial u}{\partial h} = 0, \quad \frac{\partial u}{\partial t} + \frac{1}{\rho_0}\frac{\partial P}{\partial h} = 0,$$
$$\frac{\partial v_{pore}}{\partial t} + \frac{P}{\rho_0^2 c_0^2 \tau_\mu} = 0, \quad P = \rho_0^2 c_0^2\left(\frac{1}{\rho_0} - v + v_{pore}\right), \tag{1}$$

where t is the time; h is the Lagrangian coordinate; u is the mass velocity; ρ_0 and c_0 are the initial density and velocity of sound, respectively; and τ_μ is the characteristic relaxation time of the fracture process and corresponds to bulk viscosity $\mu = \rho_0 c_0^2 \tau_\mu$.

Figure 2 shows the flow pattern in the t–h plane. In region 1 there is no interaction between the incident wave and the reflected wave, and the coordinate and time dependence of the mass velocity and pressure corresponds to a triangular compression pulse:

$$u(h,\ t) = u_0 - k(c_0 t - h),\ P(h,\ t) = \rho_0 c_0 u(h,\ t).$$

Here u_0 is the maximum mass velocity; k = const.

In region 2 the incident pulse interacts with the pulse reflected from the free surface h = 0, and this interaction results in the appearance of tensile stresses. The flow is determined by solving system (1) with boundary conditions at h = 0 and $h \to -\infty$ and initial conditions on the C_- characteristic, on which the functions under consideration, with the exception of v_{pore}, undergo a jump. We shall find the solution in the region 2. For this, we eliminate from Eqs. (1) v_{pore} and v and replace the independent variables: $T = t + h/c_0$ and $x = h$. Then the region 2 is mapped on the fourth quadrant of the T–x plane: $T \geq 0$, $x \leq 0$. Laplace transforming the obtained system of two partial differential equations in T yields a system of ordinary differential equations:

$$\frac{d\hat{u}}{dx}+\frac{s}{c_0}\hat{u}+\left(s+\frac{1}{\tau_\mu}\right)\frac{\hat{P}}{\rho_0c_0^2}=\frac{1}{\rho_0c_0^2}(P(x,0)+\rho_0c_0u(x,0)),$$
$$\frac{d\hat{P}}{dx}+\frac{s}{c_0}\hat{P}+\rho_0s\hat{u}=\frac{1}{c_0}(P(x,0)+\rho_0c_0u(x,0)) \tag{2}$$

where s is the Laplace transform variable and $\hat{u}$ and $\hat{P}$ are the Laplace transforms of the mass velocity and pressure, respectively. The initial values of u and P in the limit $T \to 0$ are transferred to the right-hand side of Eqs. (2); they appear in the form of a combination that is a Riemann invariant [6], so that it is not necessary to determine separately u and P to the right of the jump on the C_- characteristic: they will be found directly from the solution of the system of equations. The value of the invariant is found from the condition of continuity on the jump (since $v_{pore} \to 0$ as $T \to +0$) from its value in the region 1. According to Eq. (1), we obtain

$$P(x,\ 0)+\rho_0c_0u(x,\ 0)=2\rho_0c_0(u_0+2kx)\theta(x-x_0),$$

where $\theta(x)$ is the Heaviside unit function and $x_0 = h_0$ is determined from the condition of intersection of the tail C_+ characteristic and the C_- characteristic: $x_0 = -c_0\tau = -u_0/(2k)$.

The solution of the system of Eqs. (2) with the boundary conditions $\hat{P} = 0$ at $x = 0$ and $\hat{P}$ and $\hat{u}$ remain finite as $x \to -\infty$ has the form

$$\hat{P}(x,s)=-4k\rho_0c_0^2\left[\frac{\tau_\mu}{s}(\theta(x-x_0)-\exp(\lambda_1x))-\frac{1}{c_0^2(\lambda_1-\lambda_2)}\times\right.$$
$$\times\left(\frac{\theta(x-x_0)-1}{\lambda_1}\exp(\lambda_1(x-x_0))-\frac{\theta(x-x_0)}{\lambda_2}\exp(\lambda_2(x-x_0))+\right. \tag{3}$$
$$\left.\left.+\frac{1}{\lambda_2}\exp(\lambda_1x-\lambda_2x_0)\right)\right];$$

$$\hat{u}(x,s)=\frac{2k}{s}\left[2(x-x_0+c_0\tau_\mu)\theta(x-x_0)-\frac{\lambda_1-\lambda_2}{s}c_0^2\tau_\mu\exp(\lambda_1x)-\right.$$
$$-\frac{\theta(x-x_0)-1}{\lambda_1}\exp(\lambda_1(x-x_0))-\frac{\theta(x-x_0)}{\lambda_2}\exp(\lambda_2(x-x_0))-$$
$$\left.-\frac{1}{\lambda_2}\exp(\lambda_1x-\lambda_2x_0)\right], \tag{4}$$
$$\lambda_{1,2}=-s/c_0\pm\sqrt{s(s+1/\tau_\mu)}/c_0.$$

When analyzing the fracture process, it is important also to know the distribution of the specific volume of the pores, whose Laplace transform is

$$\hat{v}_{pore}=-\hat{P}/(\rho_0^2c_0^2\tau_\mu s). \tag{5}$$

Transforming from the transforms to the original functions and returning to the variables t and h, we find the solution in the fracture region. Some results, however, can be obtained by analyzing Eqs. (3) and (4) directly. For example, using the well-known property of the Laplace transform [7] $\lim_{s\to\infty} s\hat{F}(s)=F(0)$, we determine the pressure and mass velocity to the right of the jump along the C_- characteristic:

$$P=-4k\rho_0c_0^2\tau_\mu\left[\theta(h-h_0)-\exp\left(\frac{h}{2c_0\tau_\mu}\right)-(\theta(h-h_0)-1)\exp\left(\frac{h-h_0}{2c_0\tau_\mu}\right)\right]; \tag{6}$$

$$u=4kc_0\tau_\mu\left[\left(\frac{h-h_0}{c_0\tau_\mu}+1\right)\theta(h-h_0)-\exp\left(\frac{h}{2c_0\tau_\mu}\right)-(\theta(h-h_0)-1)\exp\left(\frac{h-h_0}{2c_0\tau_\mu}\right)\right]. \tag{7}$$

An analogous result for the pressure for $h \geq h_0$ was obtained in [8] in an analysis of the conditions on the jump.

The general solutions are quite unwieldy. For this reason, we investigate in greater detail the particular cases which are of greatest practical interest. It was noted above that the time dependence of the velocity of the free surface is recorded experimentally, so that it is this dependence that must be determined. Using the well-known formulas of the inverse Laplace transform and the properties of the Laplace transform itself [7, 9], we obtain the following expression for the pressure for $h \geq h_0$ and the velocity at $h = 0$:

$$P(h,t) = -4k\rho_0 c_0^2 \tau_\mu \left[1 - \exp\left(-\frac{\alpha_1}{2}\right) - \frac{\alpha_1}{2}\exp\left(-\frac{\alpha_1}{2}\right) F_1\left(\frac{t + h/c_0}{2\tau_\mu}\right) - \right.$$
$$\left. - \frac{1}{2}F_2\left(\frac{t - h/c_0 - 2\tau}{2\tau_\mu},\ \alpha_2\right)\theta\left(t - \frac{h}{c_0} - 2\tau\right) + \frac{1}{2}F_2\left(\frac{t + h/c_0 - 2\tau}{2\tau_\mu},\ \alpha_3\right)\theta(t - 2\tau)\right]; \quad (8)$$

$$u(0,t) = 2u_0 - 4kc_0\tau_\mu\left[\Phi_1\left(\frac{t}{2\tau_\mu}\right) - \Phi_2\left(\frac{t - 2\tau}{2\tau_\mu}\right)\theta(t - 2\tau)\right], \quad F_1(x) = \int_0^x \exp(-z) I_1(\sqrt{z(z+\alpha_1)})/\sqrt{z(z+\alpha_1)}\, dz,$$

$$F_2(x, \alpha) = \exp(-x - \alpha/2)\int_0^x I_0(\sqrt{z(z+\alpha)})\,[I_0(x - z) + I_1(x - z)]\,dz,$$

$$\Phi_1(x) = \exp(-x)\,[2xI_1(x) + (1 + 2x)I_0(x)] - 1, \quad (9)$$

$$\Phi_2(x) = 2\exp\left(-\frac{\tau}{2\tau_\mu}\right)\int_0^x \exp(-z) I_0(\sqrt{z(z + \tau/\tau_\mu)})\,dz - F_2(x, \tau/\tau_\mu),$$

$$\alpha_1 = -h/(c_0\tau_\mu),\ \alpha_2 = (h - h_0)/(c_0\tau_\mu),\ \alpha_3 = -(h + h_0)/(c_0\tau_\mu)$$

where I_0 and I_1 are modified Bessel functions of zeroth and first orders.

Following Eq. (5), we write the specific volume of the pores in the form

$$v_{\text{pore}}(x, T) = -\int_0^T P(x, T)\,dT/(\rho_0^2 c_0^2 \tau_\mu). \quad (10)$$

Analysis of the Solution and Comparison with Experiment. We now investigate the dependence of the velocity of the free surface on the relaxation time of the fracture process. For fixed t and $\tau_\mu \to 0$ it follows from Eq. (9) that $u(0, t) \to 2u_0$, as should be for a medium with no strength. In the second limiting case ($\tau_\mu \to \infty$) we obtain thar $u(0, t) = 2u_0 - 2kc_0 t$ for $t \leq 2\tau$ and $u(0, t) = 0$ for large t, which corresponds to motion in the absence of fracture.

We now consider the time dependence of the velocity for fixed τ_μ. Analysis of Eqs. (9) shows that at $t \to \infty$ $\mu(0, t)$ decreases monotonically, approaching zero as $1/\sqrt{t}$. At the point $t = 2\tau$ there is a break on the velocity profile, and the jump in the derivative has the form

$$\left[\frac{du(0,t)}{dt}\right]_{t=2\tau} = 2kc_0\exp\left(-\frac{\tau}{2\tau_\mu}\right). \quad (11)$$

The jump is more pronounced the longer the relaxation time. If the break in the velocity can be recorded experimentally, then the relation (11) can be used to estimate the value of τ_μ. As $t \to 0$ $u(0, t)$ touches the velocity profile which would be observed in the absence of fracture. As a result, it is impossible, in principle, to determine the fracture threshold in the case when the threshold is small and the characteristic spalling pulse is not recorded on the experimentally obtained profile.

We now consider the time dependence of the velocity for $t \leq 2\tau$. In this case, the relation (9) simplifies significantly, since the second term in brackets is equal to zero and $u(0, t)$ is expressed explicitly in terms of modified Bessel functions, tabulated values of which are presented, for example, in [9]. Velocity profiles are constructed in Fig. 3 from formula (9) in the dimensionless variables $u(0, t)/2u_0 - t/2\tau$ with $\tau_\mu/\tau = 0.01, 0.02, 0.1, 1$, and ∞ (lines 1-5). Since τ_μ determines uniquely the dependence $u(0, t)$, we consider the inverse problem: we estimate the relaxation time from the velocity at $t = 2\tau$ in the approximation $\tau_\mu \ll \tau$, which is of greatest practical interest. Using the asymptotic expansion of the modified Bessel functions for large values of the argument [9], we find

$$u(0, 2\tau) = 2u_0\left[1 - \frac{\tau_\mu}{\tau}\left(\sqrt{\frac{8}{\pi}\frac{\tau}{\tau_\mu}} - 1 + \sqrt{\frac{1}{8\pi}\frac{\tau_\mu}{\tau}} + \ldots\right)\right]. \quad (12)$$

Introducing the notation $\varepsilon = (2u_0 - u(0, 2\tau))/2u_0$ and solving Eq. (12) for τ_μ, we obtain

$$\frac{\tau_\mu}{\tau} = \frac{\pi}{8}\varepsilon^2\left[1 + \frac{\pi}{4}\varepsilon + \frac{\pi(5\pi - 2)}{64}\varepsilon^2 + \ldots\right]. \quad (13)$$

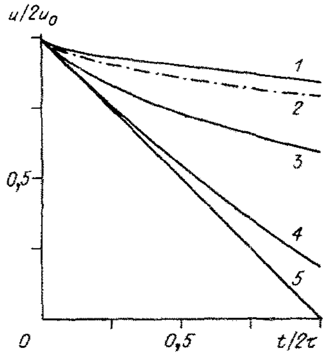

Fig. 3

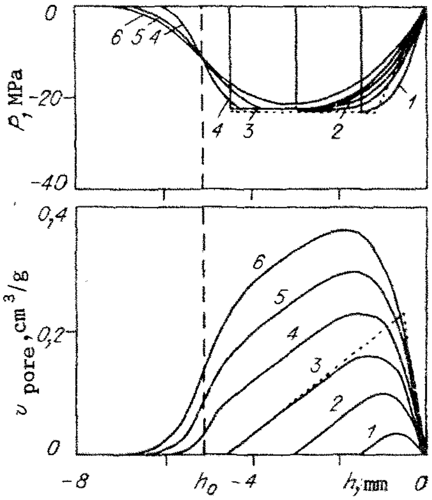

Fig. 4

The relation (13) makes it possible to determine the relaxation time directly from the experimental velocity profile. For $\varepsilon < 0.3$ the first two terms of the expansion give an accuracy of 6%.

We now investigate the pressure distribution in the fracture region. For simplicity we consider the part of the distribution which directly affects the velocity profile for $t \leq 2\tau$: $t - 2\tau - h/c_0 \leq 0$ (below the dashed line in Fig. 2). Then the last two terms in expression (8) are equal to zero. We shall show that if $\tau_\mu \ll t + h/c_0$ the pressure is practically constant in the fracture region. Indeed, the partial derivative of P with respect to time has the form

$$\frac{\partial P}{\partial t} = -2k\rho_0 c_0 h \sqrt{\frac{\tau_\mu}{\gamma}}\, \frac{\exp\left(\left(-t+\sqrt{t^2-h^2/c_0^2}\right)/2\tau_\mu\right)}{\left(t^2-h^2/c_0^2\right)^{3/4}},$$

i.e., it is exponentially small. Therefore, in the approximation under consideration the pressure is constant and is equal to its value on the jump, determined by the relation (6):

$$P \simeq -4k\rho_0 c_0^2 \tau_\mu. \quad (14)$$

Near the free surface ($|h| \ll c_0 t$) the pressure drops to zero linearly as a function of h:

$$P \simeq 4k\rho_0 c_0 h \sqrt{\tau_\mu/(\pi t)},\ \ t \gg \tau_\mu. \quad (15)$$

Using Eqs. (13) and (14), we find the following relation for the tensile stress as a function of the velocity at $t = 2\tau$:

$$P = -\frac{\pi}{4}\rho_0 c_0 u_0 \varepsilon^2 \left[1 + \frac{\pi}{4}\varepsilon + \frac{\pi(5\pi-2)}{64}\varepsilon^2 + \ldots\right]. \quad (16)$$

The relation (16) replaces the well-known formula for determining the spallation strength [1], in the sense that it makes it possible to find the tensile stresses in the fracture zone from the measured velocity profile. This, however, does not mean that the total fracture of the medium (with formation of a spall plate) will occur when the stress exceeds the threshold value. On the basis of the model examined here, it is more logical to conjecture that the material ruptures when the specific pore volume reaches its critical value.

The coordinate and time dependence of v_{pore} is given by the relation (10). In the approximation of small τ_μ, when the pressure is practically constant in the fracture region, the relation (10) implies

$$v_{\text{pore}}(h, t) \simeq (4k/\rho_0)(t + h/c_0). \quad (17)$$

Near the free surface ($|h| \ll c_0 t$) the porosity drops to zero:

$$v_{\text{pore}}(h, t) \simeq -\frac{8kh}{\rho_0 c_0^2}\sqrt{\frac{t}{\pi\tau_\mu}},\ \ t \gg \tau_\mu. \quad (18)$$

According to Eq. (17), the law of growth of the pores does not depend on the relaxation time, and it can be found directly from an analysis of the flow in a medium fracturing without resistance.

The section h^*, in which the specific pore volume reaches its maximum value, initially lies near the free surface and moves with time into the volume of the sample according to a law which can be estimated by equating expressions (17) and (18):

$$h^* \sim -c_0\sqrt{\pi\tau_\mu t}/2. \tag{19}$$

Introducing the critical value of the specific pore volume v^*_{pore}, at which the material ruptures, the corresponding time and thickness of the spall plate can then be estimated from the relations (17) and (19).

We now employ the results obtained above to describe the experimental data on the rupture of rubber under loading by a shock wave [3]. The initial density and velocity of sound are 1.34 g/cm^3 and 1.5 km/sec, respectively, and the amplitude and width of the compression pulse are u_0 = 290 m/sec and 2τ = 6.6 μsec, respectively. From the profile of the velocity of the free surface (curve 2 in Fig. 1) we obtain the value $\varepsilon = 0.21$, which makes it possible to determine, from formula (13), the characteristic relaxation time of the rupture process τ_μ = 0.068 μsec. The velocity of the free surface, calculated with these parameters from the relation (9), is presented in Figs. 1 and 3 (dashed line). One can see that the calculation is in good agreement with experiment.

The solutions (8) and (10) for the pressure and specific pore volume are quite complicated. For this reason, the system of equations of gas dynamics (1) was modeled numerically by the method of characteristics [10], using splitting according to physical processes, in order to determine P and v_{pore}. The velocity profile of the free surface, calculated in this manner, is identical to the dot-dashed line in Fig. 1. The corresponding coordinate distributions of the pressure and specific pore volume are presented in Fig. 4 with a time step of 1 μsec (the numbers indicate the times in microseconds and the dashed lines are the pressure and pore volume constructed from formulas (14), (15) and (17), (18) at t = 3 μsec). The approximations (14) and (18) describe quite accurately the maximum value of the tensile stresses and the linear nature of the increase in the specific pore volume as a function of time and the coordinate.

The amplitude of the compression pulse in the experiment with rubber is 0.9 GPa. In this pressure range it can be seen that the shock adiabatic curve is nonlinear and the acoustic approximation can lead to appreciable errors. At the same time, the tensile stresses in the fracture zone are low and it is important to know the velocity of sound under normal conditions. It can thus be expected that taking into account the pressure dependence of the velocity of sound will not produce any fundamental changes in the results. Indeed, the dashed line in Fig. 1 shows the result of the calculation of the velocity of the free surface using the equation of state constructed for rubber on the basis of the real shock adiabatic curve [3, 11]. The numerical modeling was performed using a ripple-through computational scheme on a checkerboard grid with artificial viscosity [12]. In order to obtain the best agreement with experimental data the relaxation time was reduced approximately by 10% (τ_μ = 0.06 μsec) compared with the calculation in the acoustic approximation. The distributions of the pressure and specific pore volume remain similar to those presented in Fig. 4. Only anomalies associated with the pressure dependence of the velocity of sound appear: the right-hand boundary of the fracture region moves more rapidly into the volume of the sample, and because the front of the rarefaction wave becomes diffuse the tensile stresses decrease in absolute magnitude with decreasing h and do not remain practically constant, as happens in acoustics.

Thus, on the basis of our analysis of the process of pore growth in a medium with zero fracture threshold, we have derived relations which make it possible to find the bulk fracture viscosity and the effective tensile stresses (formulas (13) and (16)) from the experimentally measured profile of the velocity of the free surface of the sample under loading by a shock wave.

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EFFECTIVE-FIELD METHOD IN THE STATICS OF COMPOSITE MATERIALS

V. A. Buryachenko and V. Z. Parton UDC 534.4

In this paper we study a linearly elastic composite medium consisting of a uniform matrix containing a random number of inclusions which have an arbitrary shape and nonuniform bulk mechanical properties. The classical problem [1-3] of estimating the effective moduli and average stress concentrators on inclusions is solved. The approach proposed in this paper is an extension of the effective-field method (EFM), presented in [4-6] for the case when the mechanical properties of the matrix are the same as those of the comparison medium. The generalized EFM includes as particular cases the well-known methods of structural mechanics: the effective-medium method [3], the generalized singular approximation method [3], the conditional moment method [7, 8], the Mori–Tanaka–Eshelby method [9, 10], and methods based on variational principles [2].

1. General Equations. Consider a macroscopic region w with characteristic function W and containing a random set $X = (V_k, x_k, \omega_k)$ of ellipsoids v_k with characteristic functions V_k and centers x_k, forming a Poisson set, semiaxes $a_k^i (a_k^1 \geq a_k^3 \geq a_k^3)$, and Euler angles ω_k. The local equation of state of the material, relating the stress tensor $\sigma(x)$ and the strain tensor $\varepsilon(x)$, is given in the form

$$\sigma(x) = L(x)\varepsilon(x), \tag{1.1}$$

where L(x), which is a tetravalent tensor of the elastic moduli, is assumed to be homogeneous in the matrix $v_0 = w \setminus v \left(v = \bigcup_{k=1} v_k\right)$: $L(x) = L^{(0)}$ in each inclusion v_k, where k = 1, 2, ..., and $L(x) = L^{(k)}(x)$ is, generally speaking, an inhomogeneous function of the coordinates. Substituting Eq. (1.1) into the equation of equilibrium with given boundary conditions on the displacements u(x), we obtain a differential equation for the displacements:

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